

Martin-Löf randomness, invariant measures and countable homogeneous structures

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Abstract

We use ideas from topological dynamics (amenability), combinatorics (structural Ramsey theory) and model theory (Fraïssé limits) to study closed amenable subgroups G of the symmetric group S_∞ of a countable set, where S_∞ has the topology of pointwise convergence. We construct G -invariant measures on the universal minimal flows associated with these groups G in, moreover, an algorithmic manner. This leads to an identification of the generic elements, in the sense of being Martin-Löf random, of these flows with respect to the constructed invariant measures. Along these lines we study the random elements of S_∞ , which are permutations that transform recursively presented universal structures into such structures which are Martin-Löf random.

Keywords: Martin-Löf randomness, topological dynamics, amenable groups, Fraïssé limits, Ramsey theory.

1 Introduction

During the past four decades there has been a vigorous development on the interplay between combinatorics and algorithmics on the one hand, and the dynamical properties of topological groups, on the other. (See, for example, Sarnak [24], as well as Lubotzky, Phillips and Sarnak [17, 18].)

Notions from the theory of topological transformation groups such as amenability or Kazhdan's property (T), play a central rôle in the construction of expander graphs (as expounded for example in [24]), which in turn are central to dealing with deterministic error amplification for the complexity class **RP** (randomized polynomial time algorithms). The aim of this very active area of research still is to minimise the number of random bits (as generated, for example, by some physical artifact) which might be required for the execution of a probabilistic computation. These results also have highly nontrivial implications for the design of quantum

circuits. (See, for example, Harrow, Recht and Chuang [10].) We are dealing here with essential instances of the deep problem of *derandomization*.

The focus of this paper is on the dual problem of understanding the symmetries that transform a recursively presented universal structure, which in this paper is a Fraïssé limit of finite first order structures, to a copy of such a structure which is Martin-Löf random relative to an S_∞ -invariant measure on the class of all universal structures of the given type. Here S_∞ is the symmetric group of a countable set, with the pointwise convergence topology.

It was shown by Kechris, Pestov and Todorcevic [14], that deep results in combinatorics (structural Ramsey theory) can be interpreted as statements on the dynamical properties of closed subgroups of S_∞ . They identified the so-called extremely amenable subgroups G of S_∞ in terms of the Ramsey properties of some classes \mathcal{F} of finite first-order structures, such that G is the symmetry group of the Fraïssé limit \mathbb{F} of the class \mathcal{F} . They also showed how one can utilise the Ramsey properties of some classes \mathcal{F} to identify the universal minimal flow \mathcal{U} of the automorphism group G of the Fraïssé limit \mathbb{F} of \mathcal{F} . It is a remarkable fact that for many interesting classes \mathcal{F} (total orders, graphs, posets, ranked diagrams, ...) the space \mathcal{U} can be embedded into a Baire space of the form $\{0,1\}^{\mathbb{N}^k}$ which renders \mathcal{U} accessible to an effective study of randomness relative to computable measures on $\{0,1\}^{\mathbb{N}^k}$ which are supported by \mathcal{U} .

In this paper we initiate a study of computable group invariant measures ν on the spaces \mathcal{U} and identify the elements of \mathcal{U} which are Martin-Löf random with respect to the measures ν . We shall study the elements of S_∞ (called randomizers in this paper) that transform universal recursive objects in \mathcal{U} to such objects which are Martin-Löf random with respect to ν . We shall also look at the recent paper of Petrov and Vershik [23] from the viewpoint of randomizing recursive universal structures.

2 Preliminaries on amenable groups

Issues involving the uniqueness of Lebesgue measure, including the Banach-Tarski paradox, led to questions of how a group G acting on a set X determines the structure (and number) of G -invariant finitely additive probability measures on X (G -invariant means). For example, Banach (1923) [1] showed that there was more than one rotation invariant finitely additive probability measure on the Lebesgue measurable subsets of S^1 . Subsequently, Sullivan (1981) [25] and independently Margulis (1980, 1981) [19] and [20] (for $n \geq 4$) and Drinfeld (1984) [4] (for $n = 2, 3$) showed that Lebesgue measure is the unique finitely additive rotation invariant measure on the Lebesgue measurable subsets of S^n . (It is crucial to consider the Lebesgue measurable sets here. The corresponding problem for Borel-measurable sets is still open.) It is probably fair to say that the entire development of amenable groups arose from this interest.

Let G be a topological group and X a compact Hausdorff space. A dynamical system (X, G) (or a G -flow on X) is given by a jointly continuous action of G on X . If (Y, G) is a second dynamical system, then a G -morphism $\pi : (X, G) \rightarrow (Y, G)$ is

a continuous mapping $\pi : X \rightarrow Y$ which intertwines the G -actions, i.e., the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ 1 \times \pi \downarrow & & \downarrow \pi \\ G \times Y & \xrightarrow{\beta} & Y \end{array},$$

commutes with α, β being the group actions.

An isomorphism is a bijective homomorphism. A subflow of (X, G) is a G -flow on a compact subspace Y of X with the action the restricted to the action of G on X to the action on Y . The dynamical system (X, G) is said to be *minimal* if every G -orbit is dense in X . Equivalently, a G -flow is minimal if it has no proper subflows. Every dynamical system has a minimal subflow (Zorn).

The following fact, first proven by Ellis (1949) [5], is central to the theory of dynamical systems:

Theorem 1 *Let G be a Hausdorff topological group. There exists, up to G -isomorphism, a unique minimal dynamical system, denoted by $(M(G), G)$, such that for every minimal dynamical system (X, G) there exists a G -epimorphism $\pi : (M, G) \rightarrow (X, G)$, and any two such universal systems are isomorphic.*

The flow $(M(G), G)$ is called the *universal minimal flow* of G . We next introduce the notion of amenable groups.

Definition 1 *A topological group G is amenable if, whenever X is a non-empty compact Hausdorff space and π is a continuous action of G on X , then there is a G -invariant Borel probability measure on X .*

This means that, for every G -flow on a compact space X , there is a measure ν on the Borel algebra of X , such that, $\nu(X) = 1$ and, for every $g \in G$ and Borel subset U of X ,

$$\nu(gU) = \nu(U).$$

It follows that G is amenable iff its universal minimal flow $M(G)$ has a G -invariant probability measure. Indeed, let ν be an invariant measure on $M(G)$. Consider any G -flow on some compact Hausdorff space X . By Zorn's lemma there is a minimal subflow Y and a G -embedding i of Y into X . Therefore, there are G -morphisms

$$M(G) \xrightarrow{\pi} Y \xrightarrow{i} X.$$

Let ρ be the pushout measure of ν under $i\pi$. In other words, for every Borel subset A of X , we set

$$\rho(A) = \nu(\pi^{-1}i^{-1}A).$$

Then ρ is an invariant measure on X . The converse is trivial, since $M(G)$ is a compact G -flow.

If G is compact, then it is well-known that $M(G) \simeq G$, topologically, and the Haar measure is therefore the (unique) invariant measure on $M(G)$. In particular,

a compact Hausdorff group is amenable. It can be shown that a discrete group G is amenable (see, for example [8]) iff it admits a G -invariant finitely additive probability measure on all the subsets of G . It is a classical fact that the free group \mathbb{F}_2 on two elements is not amenable; this result played an important rôle in understanding the Banach-Tarski paradox.

We make frequent use of the following:

Theorem 2 *Let G be a topological group and suppose there is a dense subset of G such that every finite subset of G is included in an amenable subgroup of G , then G too is amenable.*

A proof of this theorem can be found in Section 449C of [8].

For notational convenience, we shall assume in the sequel that the elements of S_∞ are permutations of the natural numbers \mathbb{N} . There is natural embedding of S_∞ into the space $\mathbb{N}^\mathbb{N}$. We give $\mathbb{N}^\mathbb{N}$ the product topology (with \mathbb{N} having the discrete topology). We topologise S_∞ with the inherited topology from $\mathbb{N}^\mathbb{N}$. For obvious reasons, this is called the topology of pointwise convergence.

For a natural number n and two n -tuples (a_1, \dots, a_n) and (b_1, \dots, b_n) of natural numbers, each n -tuple having n distinct elements, we denote by

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

the set of elements of S_∞ that map a_i to b_i for all $i = 1, \dots, n$. Each of these sets is open in S_∞ and all these (cylindrical) sets constitute a basis for the topology on S_∞ .

Since any finite group (with the discrete topology), being compact, is amenable, it follows from Theorem 2 that if a group contains a dense locally finite subgroup, then it is amenable. (Recall that a group is locally finite if every finite subset of the group generates a finite group.) In particular, any locally finite group is amenable. Let S_0 be the subgroup of S_∞ consisting of all the permutations that fix all but a finite set of elements. Then S_0 is dense in S_∞ and the former is obviously locally finite. It follows that S_∞ is an amenable group.

In contrast to locally compact groups, it is readily seen that not all closed subgroups of S_∞ are amenable groups. Indeed, a Caley embedding would embed any countable group as a closed discrete subgroup of S_∞ . Indeed, let G be a countable group. The Caley embedding c is defined to be

$$c : G \longrightarrow S_\infty,$$

given by

$$\sigma \mapsto (\sigma),$$

where (σ) is the permutation on \mathbb{N} given by $x \mapsto \sigma x$, for all $x \in \mathbb{N}$. It is clear that the image of G under c inherits from S_∞ the discrete topology. Moreover, $c(G)$ is easily seen to be closed in S_∞ . (In fact, any discrete subgroup in a Hausdorff topological group is necessarily closed!)

In particular, the non-amenable group \mathbb{F}_2 can be embedded as a discrete and closed subgroup of S_∞ .

We next discuss a beautiful example of Pestov [22]: Let η be the countable linear order, called the rational order, which is order isomorphic to the standard ordering of the rational numbers. Let A be the automorphism group of the ordering η . Then A is a closed subgroup of S_∞ . As has already been mentioned, the free group on two elements \mathbb{F}_2 with the discrete topology is not an amenable group. Let $(D, <)$ be a countable linearly ordered skew field. Then the multiplicative group D^\times of D acts via left-multiplication on the total order $<$.

However, as shown by Neumann (1949) [21], the order $<$ is isomorphic to η . In this way one embed D^\times as a closed and discrete subgroup of A .

Finally, Neumann also showed that \mathbb{F}_2 can be embedded into D^\times . In this way we get a closed embedding of \mathbb{F}_2 in A .

This construction is extremely interesting, for the group A is in fact extremely amenable (no pun intended). This means that every continuous action of A on a compact metric space has a fixed point.

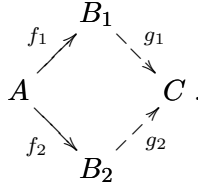
In [2] Bhattacharjee and Macpherson have shown that there is a locally finite dense subgroup of the symmetry group of the random graph, thus allowing us to infer that the symmetry group of the well-known random graph is amenable. We shall later show how one can algorithmically construct an invariant mean on the universal minimal flow of the symmetry group of the random graph.

It was recently shown by Kechris and Soliř [16] that the symmetry group of the Fraïssé limit (see the following section for definitions) of finite posets is not amenable. The Ramsey properties of finite posets (see, for example, [6]) play an important role in their argument. These results will later be used in this paper to show that the set of linear extensions of the Fraïssé limit of finite posets are all, in a definite sense, nonrandom, at least from the point of view of algorithmic randomness.

3 Fraïssé limits and their recursive representations

In the sequel, \mathcal{L} will stand for the signature of a relational structure. Moreover, \mathcal{L} will always be finite and the arities of the relational symbols will all be ≥ 1 . The definitions that follow were introduced by Fraïssé in 1954.

The *age* of an \mathcal{L} -structure X , written $\text{Age}(X)$, is the class of all finite \mathcal{L} -structures (defined on finite ordinals) which can be embedded as \mathcal{L} -structures into X . The structure X is *homogeneous* (some authors say *ultrahomogeneous*) if, given any isomorphism $f : A \rightarrow B$ between finite substructures of X , there is an automorphism g of X whose restriction to A is f . A class \mathbf{K} of finite \mathcal{L} -structures has the *amalgamation property* if, for structures A, B_1, B_2 in \mathbf{K} and embeddings $f_i : A \rightarrow B_i$ ($i = 1, 2$) there is a structure C in \mathbf{K} and there are embeddings $g_i : B_i \rightarrow C$ ($i = 1, 2$), such that the following diagram commutes:



Suppose \mathbf{K} is a countable class of finite \mathcal{L} -structures, the domains of which are finite ordinals such that

1. if A is a finite \mathcal{L} -structure defined on some finite ordinal, if $B \in \mathbf{K}$ and if there is an embedding of A into B , then $A \in \mathbf{K}$;
2. the class \mathbf{K} has the amalgamation property.

Then, Fraïssé showed that there is a countable homogeneous structure X such that $\text{Age}(X) = \mathbf{K}$. Moreover, X is unique up to isomorphism. The (essentially) unique X is called the *Fraïssé limit* \mathbb{K} of \mathbf{K} . Note that, conversely, the age \mathbf{K} of a countable homogeneous structure has properties (1) and (2). We shall frequently call a countable structure which is isomorphic to a Fraïssé limit a *universal structure*.

A *recursive representation* of a countably infinite \mathcal{L} -structure X is a bijection $\phi : X \rightarrow \mathbb{N}$ such that, for each $R \in \mathcal{L}$, if the arity of R is n , then the relation R^ϕ defined on \mathbb{N}^n by

$$R^\phi(x_1, x_2, \dots, x_n) \iff R(\phi^{-1}(x_1), \dots, \phi^{-1}(x_n)),$$

is recursive. If we identify the underlying set of X with \mathbb{N} via ϕ and each R with R^ϕ , we call the resulting structure a *recursive \mathcal{L} -structure* on \mathbb{N} and we say it has a recursive representation on \mathbb{N} .

If X is countable and homogeneous and if $\text{Age}(X)$ has an enumeration A_0, A_1, A_2, \dots , possibly with repetition, with the property that there is a recursive procedure that yields, for each $i \in \mathbb{N}$, and $R \in \mathcal{L}$, the underlying set $A(i)$ of A_i together with the interpretation of R in $A(i)$, then we call $(A_i : i \in \mathbb{N})$ a recursive enumeration of $\text{Age}(X)$. It follows from the construction of Fraïssé limits from their ages, that one can construct a recursive representation of X from a recursive enumeration of its age. (Conversely, it is trivial to derive a recursive enumeration of $\text{Age}(X)$ from a recursive representation of X .) It is therefore not difficult to find recursive representations for Fraïssé limits of classes \mathbf{K} from recursive enumerations of their ages.

Theorem 3 *Suppose \mathbb{C} and \mathbb{D} are countable recursively represented \mathcal{L} -structures on \mathbb{N} with the same age. Suppose that they are both homogeneous. Then there is a recursive isomorphism from \mathbb{C} to \mathbb{D} .*

Proof. The model-theoretic back-and-forth argument as discussed, for example, on pp 161-162 of Hodges [11] is constructive relative to the recursive representations of the homogeneous structures \mathbb{C} and \mathbb{D} .

Let \mathcal{K} be a Fraïssé class of finite structures. We say that \mathcal{K} has the *Hrushovski property* if for any A in \mathcal{K} there is B in \mathcal{K} containing A such that any partial automorphism of A extends to an automorphism of B . The terminology, due to Kechris

and Rosendal [15], is probably inspired by the result by Hrushovski [12] who established that the class of finite graphs has this property. In [15] it is shown that a Fraïssé class of finite structures \mathcal{K} has the Hrushovski property iff the automorphism group G of the Fraïssé limit of \mathcal{K} is compactly approximable, i.e., there is a increasing sequence K_n of compact subgroups whose union is dense in the automorphism group. Since a compact group is amenable, it follows from Theorem 2 that the automorphism group of the Fraïssé limit of a Fraïssé class with the Hrushovski property will be amenable.

4 Martin-Löf random countable orders

Let S_∞ be the group of permutations of a countable set, which, without loss of generality, we may take to be \mathbb{N} . We place on S_∞ the pointwise convergence topology.

Let $(\mathbb{N} \times \mathbb{N})_\neq$ denote the set of ordered pairs (i, j) of natural numbers with $i \neq j$. Write \mathcal{M} for the set of total orders on \mathbb{N} . We identify \mathcal{M} with a subset of $\{0, 1\}^{(\mathbb{N} \times \mathbb{N})_\neq}$ by identifying a total order $<$ on \mathbb{N} with the function $\xi : (\mathbb{N} \times \mathbb{N})_\neq \rightarrow \{0, 1\}$ given by

$$\xi(x, y) = 1 \Leftrightarrow x < y, \quad x, y \in \mathbb{N}.$$

The total order associated with ξ will be denoted by $<_\xi$. We topologise \mathcal{M} via the natural injection $\mathcal{M} \rightarrow \{0, 1\}^{(\mathbb{N} \times \mathbb{N})_\neq}$, where the (Baire) space $\{0, 1\}^{(\mathbb{N} \times \mathbb{N})_\neq}$ has the product topology. As such \mathcal{M} is a closed hence compact subspace of $\{0, 1\}^{(\mathbb{N} \times \mathbb{N})_\neq}$.

For each finite order l on a subset of \mathbb{N} , write A_l for the space of total orders ξ on \mathbb{N} which extends l . Then the class (A_l) with l ranging over all the finite total orders on subsets of \mathbb{N} is an open basis of neighbourhoods of \mathcal{M} . The group S_∞ acts continuously on \mathcal{M} if, for $\xi \in \mathcal{M}$ and $\sigma \in S_\infty$, we define the total order $\sigma\xi$ by:

$$x <_{\sigma\xi} y \iff \sigma^{-1}x <_\xi \sigma^{-1}y, \quad x, y \in \mathbb{N}.$$

It follows from the classical (finitary) Ramsey theorem and the fundamental paper by Kechris, Pestov and Todorćević (2005) that the group action

$$S_\infty \times \mathcal{M} \rightarrow \mathcal{M},$$

$$(\sigma, \xi) \mapsto \sigma\xi$$

is indeed isomorphic to the universal minimal S_∞ -flow. The original proof of this result can be found in the paper [9] of Glasner and Weiss (2002). Their proof is also based on the classical Ramsey theorem.

Since S_∞ is an amenable group, there is an S_∞ -invariant (Borel regular) probability measure on \mathcal{M} . In fact, Glasner and Weiss (2002) [9] showed that there is *exactly one* such measure (i.e., the flow on \mathcal{M} is uniquely ergodic). Their proof is based on an ergodic argument.

Let us denote this measure by μ . The author believes this beautiful measure deserves to be called the *Glasner-Weiss measure*.

In this paper, we wish to understand the μ -Martin-Löf random elements of \mathcal{M} . This is a viable project, for it will now be shown, with the benefit of hindsight,

that the Glasner-Weiss measure can be computed and effectively constructed. (For more on the subject of Martin-Löf randomness relative to computable measures, the reader is referred to the recent survey paper [3] together with the many references to be found there.)

We write \mathcal{M}_f for the set of finite total orders on some subset of \mathbb{N} . For $\ell \in \mathcal{M}_f$, denote by Z_ℓ the set of $\xi \in \mathcal{M}$, such that ξ is an extension of ℓ . These sets are the cylinder subsets of \mathcal{M} . Write \mathcal{Z}_0 for the class of events of the form Z_ℓ for some $\ell \in \mathcal{M}_f$ and \mathcal{Z} for the algebra generated by \mathcal{Z}_0 . Note that the σ -algebra generated by \mathcal{Z} is exactly the Borel algebra on \mathcal{M} .

Clearly, for $\sigma \in S_\infty$ and $\ell \in \mathcal{M}_f$,

$$\sigma Z_\ell = Z_{\sigma\ell}.$$

Furthermore, writing S_ℓ for the subgroup of S_∞ each element of which permutes the underlying set of ℓ and leaves all the other elements of \mathbb{N} fixed, we have the following partition:

$$\mathcal{M} = \bigsqcup_{\sigma \in S_\ell} \sigma Z_\ell.$$

Since S_∞ is amenable and \mathcal{M} is compact, there is an S_∞ -invariant probability measure ν on the Borel subsets of \mathcal{M} . For such a measure, it must follow

$$1 = \nu(\mathcal{M}) = \sum_{\sigma \in S_\ell} \nu(\sigma Z_\ell).$$

Since ν is S_∞ -invariant, it follows that

$$\nu(Z_\ell) = \frac{1}{l!}, \tag{1}$$

where l is the cardinality of the underlying set of ℓ . Since a Borel measure on \mathcal{M} is uniquely determined by its values on \mathcal{Z}_0 , the uniqueness of the invariant mean ν has been established. In particular, ν is the Glasner-Weiss measure μ .

For $Z \in \mathcal{Z}_0$ we write Z^0 for the complement of Z and Z^1 for Z . (This peculiar notation has been designed in order to justify an algorithm for computing μ on \mathcal{Z} , as will become clear in the sequel.) Let $(T_i)_{i \in \mathbb{N}}$ be any enumeration of the algebra \mathcal{Z} generated by $(Z_\ell)_{\ell \in \mathcal{M}_f}$ in such a way that one can effectively retrieve from a given $i \in \mathbb{N}$, the corresponding T_i as a finite union of sets T of the form

$$T = Z_{\ell_1}^{\delta_1} \cap \dots \cap Z_{\ell_k}^{\delta_k}, \tag{2}$$

where each ℓ_i is in \mathcal{M}_f and $\delta_i \in \{0, 1\}$ for $i = 1, \dots, k$. We call any such enumeration a *recursive representation* of \mathcal{Z} .

The measure μ is computable in the following sense:

Theorem 4 *Denote by μ the Glasner-Weiss measure on the Borel-algebra of \mathcal{M} . Let $(T_i : i < \omega)$ be a recursive representation of the algebra \mathcal{Z} . There is an effective procedure that yields, for $i, k \in \mathbb{N}$, a binary rational β_k such that*

$$|\mu(T_i) - \beta_k| < 2^{-k}.$$

Proof. By the principle of inclusion and exclusion, it suffices to find a recursive procedure for computing the μ -measure of all sets of the form (2). For such an T , we call $n - (\delta_1 + \dots + \delta_n)$ the *weight* of the representation. It follows from (1) that we can compute $\mu(T)$ up to arbitrary accuracy for all T having a representation of weight 0. Now if T has a representation of weight $f+1$, say, we can effectively write T as $T' \cap T_i^0$ where $T_i \in \mathcal{Z}_0$ and T' has a representation of weight f . However,

$$\mu(T') = \mu(T' \cap T_i^0) + \mu(T' \cap T_i),$$

so we can recursively call the procedure for computing any $\mu(G)$ with G having a representation of weight f , to compute $\mu(T')$ and $\mu(T' \cap T_i)$ up to accuracy $2^{-(k+1)}$ and hence to compute $\mu(T)$ up to accuracy 2^{-k} .

Definition 2 A set $A \subset \mathcal{M}$ is of constructive measure 0, if, for some recursive representation of $(T_i : i \in \mathbb{N})$ of \mathcal{Z} , there is a total recursive $\phi : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that

$$A \subset \bigcap_n \bigcup_m T_{\phi(n,m)}$$

and $\mu(\bigcup_m T_{\phi(n,m)})$ converges effectively to 0 as $n \rightarrow \infty$.

Definition 3 A total order ξ is said to be μ -Martin-Löf random if ξ is in the complement of every subset B of \mathcal{M} of constructive measure 0.

Write $ML_\mu \subset \mathcal{M}$ for the set of μ -Martin-Löf random total orders. Note that $\mu(ML_\mu) = 1$. We now prove the following

Theorem 5 Write \mathcal{Q} for the set of total orders on \mathbb{N} which are isomorphic to the rational order η . Then

$$ML_\mu \subset \mathcal{Q}.$$

In particular,

$$\mu(\mathcal{Q}) = 1.$$

We first introduce a number of (standard) recursion-theoretic concepts and terminology: A sequence (A_n) of sets in \mathcal{Z} is said to be *semirecursive* if for each n , the set A_n is of the form $T_{\phi(n)}$ for some total recursive function $\phi : \omega \rightarrow \omega$ and some effective enumeration (T_i) of \mathcal{Z} . (Note that the sequence (A_n^c) , where A_n^c is the complement of A_n , is also an \mathcal{Z} -semirecursive sequence.) In this case, we call the union $\bigcup_n A_n$ a \sum_1^0 set. A set is a \prod_1^0 set if it is the complement of a \sum_1^0 set. It is of the form $\bigcap_n A_n$, for some \mathcal{Z} -semirecursive sequence (A_n) . The proof of Theorem 5 is based on the following

Proposition 1 If A is a \sum_1^0 subset of \mathcal{M} and if $\mu(A) = 1$, then ML_μ is contained in A . In particular, if B is a Π_1^0 subset of \mathcal{M} that contains some element of ML_μ , then $\mu(B) > 0$.

Proof. Let P be the complement of A . We can write $P = \bigcap_n P_n$ for some semirecursive sequence (P_n) . By replacing each P_n by $\bigcap_{k < n} P_k$, the sequence remains semirecursive but now P_n is decreasing in n . Since $\lim_{n \rightarrow \infty} \mu(P_n) = 0$ it follows, from the computability of μ , that, for each n one can effectively find some $f(n)$ such that $\mu(P_{f(n)}) < (n+1)^{-1}$. Since $\mu(P_i)$ is decreasing in i , we then have that $\mu(P_m) < (n+1)^{-1}$ for all $m \geq f(n)$. We conclude that P is of constructive measure 0.

Proof of Theorem 5. In view of Cantor's well-known first-order characterisation of the rational order, it will suffice to proof the following: If $\xi \in ML_\mu$, then, for all natural numbers n, m with $n \neq m$,

$$\exists_j n <_\xi j <_\xi m; \quad (3)$$

moreover, for every natural number n

$$\exists_{j,k} j <_\xi n <_\xi k. \quad (4)$$

Note that, for fixed n, m , both of these predicates are Σ_1^0 in ξ . It follows from Proposition 1 that Theorem 5 will follow once we have shown that the predicates (3) and (4) define events both of which have μ -measures 1.

Write $C_{n,m}$ for the complement of the event defined by the predicate (3). Note that

$$C_{n,m} = \bigcap_N C_{n,m}^N,$$

where

$$\xi \in C_{n,m}^N \Leftrightarrow \forall_{j \leq N} j \leq_\xi n \vee j \geq_\xi m.$$

It follows from (1) that, for $N > n, m$, we have

$$\mu(C_{n,m}^N) = \frac{L}{N!},$$

where L is the number of total orders ℓ on $\{1, \dots, N\}$ such that n, m are adjacent with respect to the total order ℓ . Clearly

$$L = 2(N-1)(N-2)!.$$

Consequently,

$$\mu(C_{n,m}) = \lim_{N \rightarrow \infty} \mu(C_{n,m}^N) = 0.$$

Hence, if $\xi \in ML_\mu$, then ξ satisfies (3) for all n, m with $n \neq m$. The proof that ξ will satisfy (4) for all n is similar. This concludes the proof of Theorem 5.

Remark: By using standard recursion-theoretic techniques it is not difficult to show that ML_μ is in fact Σ_2^0 with a complement being a set of constructive measure 0. In other words, there is a *universal* μ -Martin-Löf random test.

Let $\mathbb{P} = (P, \prec)$ be the universal countable poset. This is, by definition, the Fraïssé limit of the class of finite posets. If A and B are subsets of P , then we write $A \prec B$, if for all $a \in A$ and $b \in B$, it is the case that $a \prec b$. The structure \mathbb{P} has the following first order characterisation:

If A, B, Z are finite and pairwise disjoint subsets of the underlying set P of \mathbb{P} such that $A \prec B$ and, for all $a \in A, b \in B$ and $z \in Z$:

$$\neg z \prec a, \neg b \prec z,$$

then there is some element x in \mathbb{P} such that $A \prec x \prec B$ and x is incomparable with all the elements of Z .

Let \mathcal{P}^* be the class of all structures of the form $(\pi, <)$, where π is a finite poset (with underlying set some ordinal), and where $<$ is a total order on the underlying set of π which is, moreover, a linear extension of π . Then \mathcal{P}^* is a Fraïssé class and its limit is of the form $\mathbb{P}^* = (\mathbb{P}, <)$, where \mathbb{P} is the universal poset and $<$ is some linear extension of \mathbb{P} . We call the linear extension $<$ of \mathbb{P}^* the *canonical* linear extension of \mathbb{P} . It readily follows from the fact that the structure \mathbb{P}^* is homogeneous, that the canonical linear extension $<$ is isomorphic to the countable rational order η . To see this, let x, y be elements of the universal poset such that $x < y$ with respect to the canonical linear extension. Choose elements x_1, z_1, y_1 in the universal poset \mathbb{P} such that $x_1 < z_1 < y_1$, and such that the induced \mathbb{P}^* -structure $X_1 = ((x_1, y_1) \prec, <)$ is isomorphic to $X = ((x, y) \prec, <)$. Let π_1 be an isomorphism from X_1 to X and use the homogeneity of \mathbb{P}^* to find an automorphism π which extends π_1 . Setting $z = \pi(z_1)$, we have found an element z such that $x < z < y$. In a similar way, we can find, for every element x some elements y and z , such that $z < x < y$. It follows that $<$ is a rational total order.

In the sequel, we shall fix a recursive representation of $(\mathbb{P}, <)$. This means that we can view the underlying set of \mathbb{P} as the natural numbers with both the relations \prec and $<$ being decidable. Write $\mathcal{M}(\mathbb{P})$ for the set of linear extensions of \mathbb{P} . Clearly $\mathcal{M}(\mathbb{P})$ is a closed subspace of \mathcal{M} .

The automorphism group $\text{Aut}(\mathbb{P})$ acts naturally on the space $\mathcal{M}(\mathbb{P})$. Note that the orbit of the canonical linear extension of \mathbb{P} under the action of $\text{Aut}(\mathbb{P})$ is contained in the class \mathcal{Q} of rational orders on the natural numbers \mathbb{N} . It turns out that all the elements in this orbit are not μ -Martin-Löf random.

Theorem 6 *Fix a recursive representation of the universal poset \mathbb{P} on the natural numbers \mathbb{N} . Let $\mathcal{M}(\mathbb{P})$ be the class of linear extensions of \mathbb{P} . Write ML_μ for the set of total orders on \mathbb{N} that are Martin-Löf random relative to the Glasner-Weiss probability measure μ . Then*

$$ML_\mu \cap \mathcal{M}(\mathbb{P}) = \emptyset.$$

Proof: Note that for $\xi \in \mathcal{M}$, it is the case that

$$\xi \in \mathcal{M}(\mathbb{P}) \iff \forall_{x \in \mathbb{N}} \forall_{y \in \mathbb{N}} (x \prec y \Rightarrow x <_\xi y),$$

which, since \prec is recursive over \mathbb{N} , means that $\mathcal{M}(\mathbb{P})$ is a Π_1^0 -subset of \mathcal{M} . By Proposition 1, if it were the case that $ML_\mu \cap \mathcal{M}(\mathbb{P}) \neq \emptyset$, then $\mu(\mathcal{M}(\mathbb{P})) > 0$. However, as has been noted before, Kechris and Sokič [16] has recently shown that the automorphism group $\text{Aut}(\mathbb{P})$ is *not* amenable. Moreover, it is shown by Kechris

et al in [14], that the structural Ramsey theory of finite posets (see, for example, [6]) has the implication that this $\text{Aut}(\mathbb{P})$ -flow on $\mathcal{M}(\mathbb{P})$ is in fact isomorphic to the universal minimal $\text{Aut}(\mathbb{P})$ -flow; allowing us to infer that $\mu(\mathcal{M}(\mathbb{P})) = 0$. This concludes the proof of the theorem.

For a rational order η , set

$$S_\mu(\eta) := \{\sigma \in S_\infty : \sigma\eta \in ML_\mu\}.$$

By Theorem 5, if η were not rational, the corresponding set $S_\mu(\eta)$ would have been the empty set. If η is rational, then the class \mathcal{Q} is exactly the orbit of η under the action of S_∞ . Since both \mathcal{Q} and ML_μ have μ -measure one, it follows that

$$\mu(\mathcal{Q} \cap ML_\mu) = 1,$$

and, therefore, that $S_\mu(\eta) \neq \emptyset$. Note that, if $\pi \in S_\infty$, then

$$S_\mu(\eta)\pi^{-1} = S_\mu(\pi\eta). \quad (5)$$

Indeed, for $\alpha \in S_\mu(\eta)$, we have $\alpha\pi^{-1}(\pi\eta) = \alpha\eta \in ML_\mu$ and hence $\alpha\pi^{-1} \in S_\mu(\pi\eta)$. Conversely, if $\tau \in S_\mu(\pi\eta)$, then $\tau\pi\eta \in ML_\mu$, i.e., $\tau\pi \in S_\mu(\eta)$, and, so, $\tau \in S_\mu(\eta)\pi^{-1}$.

If $\eta_1, \eta_2 \in \mathcal{Q}$, there is some $\pi \in S_\infty$ such that $\eta_2 = \pi\eta_1$. Moreover, if η_1, η_2 were both recursive, the permutation π could also be chosen to be recursive. (See Theorem 3). Write S_r for the class of recursive permutations of \mathbb{N} . We let S_r act on the right on the class Σ of all sets of the form $S_\mu(\tau)$ with τ a recursive rational order on \mathbb{N} . The action is given by

$$\Sigma \times S_r \longrightarrow \Sigma,$$

$$(S_\mu(\tau), \pi) \mapsto S_\mu(\tau)\pi^{-1}, \quad \pi \in S_r, \tau \in \mathcal{Q}_r,$$

where \mathcal{Q}_r denotes the class of all recursive rational orders on \mathbb{N} . It follows from the preceding arguments that this S_r -action will have a single orbit, i.e., the action is transitive. Set

$$\mathcal{S} = \bigcup_{\tau \in \mathcal{Q}_r} S_\mu(\tau).$$

If we choose any fixed $\eta \in \mathcal{Q}_r$, we also have

$$\mathcal{S} = \bigcup_{\pi \in S_r} S_\mu(\eta)\pi^{-1}.$$

We shall call the permutations in \mathcal{S} *Martin-Löf randomizers*. These are the permutations that transform some recursive rational order to one which is μ -Martin-Löf random.

If $\pi \in \text{Aut}(\mathbb{P})$ and η is a recursive rational linear extension of \mathbb{P} , then $\pi\eta$ is also a linear extension of \mathbb{P} , so, by Theorem 6, we have $\pi \notin S_\mu(\eta)$. In particular,

Theorem 7 *No Martin-Löf randomizer of a recursive rational linear extension of the universal poset \mathbb{P} can be an automorphism of \mathbb{P} .*

Let \mathbb{G} be the Fraïssé limit of finite graphs. It follows, again from [14], that the universal minimal flow of the automorphism group G of \mathbb{G} is isomorphic to the space \mathcal{M} of total orders with group action being the restriction of the S_∞ -action to G . Of course the Glasner-Weiss measure μ is also G -invariant, which gives another proof of the amenability of G . Kechris [13] has recently announced that he, in collaboration with Angel and Lyon, have recently shown that μ is the only G -invariant probability measure on G . Apparently, the fact that graphs are Hrushovski-structures [12] plays an important rôle in the proof. It would be interesting to understand which symmetries of the graph \mathbb{G} are in fact Martin-Löf randomizers.

Write $\mathbb{N}^{[2]}$ for the set of two element subsets of \mathbb{N} . We can use $\mathcal{G} = \{0, 1\}^{\mathbb{N}^{[2]}}$ as a representation of all countable graphs (with the product topology) with underlying set \mathbb{N} . The group S_∞ acts continuously on \mathcal{G} , if, for $\pi \in S_\infty$ and $\alpha \in \mathcal{G}$, we set

$$\pi\alpha(\{i, j\}) = \alpha(\{\pi^{-1}i, \pi^{-1}j\}),$$

for all $i, j \in \mathbb{N}$. We shall refer to this action as the canonical S_∞ -flow on \mathcal{G} .

Write $\mathcal{R} \subset \mathcal{G}$ for the set of $\alpha \in \mathcal{G}$ corresponding to the copies of \mathbb{G} , the Fraïssé limit of finite graphs. It is well-known that a graph $\alpha \in \mathcal{G}$ defines a copy of \mathbb{R} , iff, whenever A, B are disjoint finite subsets of \mathbb{N} , there is some z in the complement of $A \cup B$ such that $\alpha(\{z, a\}) = 1$ for all $a \in A$, and $\alpha(\{z, b\}) = 0$ for all $b \in B$. This result has the following consequence: If $\pi \in S_\infty$ and $\alpha \in \mathcal{R}$, then $\pi\alpha \in \mathcal{R}$. This means that S_∞ acts naturally on the class \mathcal{R} of random graphs.

Let λ be the Lebesgue measure on $\mathcal{N} = \{0, 1\}^{\mathbb{N}}$ and write ML_λ for λ -Martin-Löf elements of $\{0, 1\}^{\mathbb{N}}$. Let ϕ be any recursive bijection from $\mathbb{N}^{[2]}$ to \mathbb{N} . Then ϕ induces a recursive isomorphism $f := \{0, 1\}^\phi$ from \mathcal{N} to \mathcal{G} . In this way, we find a recursive embedding of ML_λ into \mathcal{R} , the latter being all countable graphs isomorphic to \mathbb{G} . (See Fouché and Potgieter [7].) Write λ_1 for the pushout of λ under f . Then f induces a recursive isomorphism between ML_λ and ML_{λ_1} . These are all the random graphs encoded via ϕ by some infinite binary string which is λ -Martin-Löf random. Note that λ_1 is a S_∞ -invariant measure under the canonical S_∞ -flow.

In recent work, Petrov and Vershik (2010) [23] have made a systematic study of the invariant measures relative the canonical S_∞ -flow restricted to the class \mathcal{R} . They have identified a continuum of S_∞ -invariant measures on \mathcal{R} . The study of Martin-Löf randomness and the associated randomizers relative to the computable S_∞ -invariant measures, is at present under investigation by the author.

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